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Newton-like methods for integral
equations with semidegenerate
kernels.

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ABSTRACT

Linear Fredholm integral equations of the second kind with semidegenerate kernels are formulated so that the numerical integration approximation gives a system of linear equations in which the matrix has a band structure. This formulation is very well suited for iterative solution of nonlinear problem. A class of Newton-like methods, which fill in a gap between the method of successive approximations and Newton's method, is given.

1. INTRODUCTION

Consider the family of integral equations

$$(1) \quad y(x) - \int_0^1 K(x,t)y(t)dt = g(x), \quad 0 \leq x \leq 1,$$

where the kernel K and the inhomogeneous term g are given, and the function y is to be determined for $0 \leq x \leq 1$.

We assume that the kernel has the form

$$(2) \quad K(x,t) = \begin{cases} \sum_{\mu=1}^M a_{\mu}(x)b_{\mu}(t), & x > t, \\ \sum_{\nu=1}^N c_{\nu}(x)d_{\nu}(t), & x < t. \end{cases}$$

We call K a semidegenerate kernel. Approximation of the integral in equation (1) by a quadrature formula gives a linear algebraic system to solve. The matrix of this system is dense.

Our aim is to transform the given integral equation into a form which after numerical integration approximation gives a linear system in which the matrix has a simple band structure.

Green's function for two-point boundary value problems can be written in the form (2). For second order differential equations Newton-like methods have been studied in Aulin [2], [3].

2. DERIVATION OF THE BAND STRUCTURE

Suppose that $[x-Mh, x+Nh] \subset [0, 1]$. By equations (1) and (2) we have, for $j = -M, \dots, 0, \dots, N$,

$$\begin{aligned} y(x+jh) - \int_0^{x+jh} \sum_{\mu=1}^M [a_{\mu}(x+jh)b_{\mu}(t)] y(t) dt + \\ + \int_{x+jh}^1 \sum_{\nu=1}^N [c_{\nu}(x+jh)d_{\nu}(t)] y(t) dt = g(x+jh), \end{aligned}$$

which we rewrite as

$$\begin{aligned} y(x+jh) - \sum_{\mu=1}^M a_{\mu}(x+jh) \int_0^{x-Mh} b_{\mu}(t) y(t) dt - \\ - \sum_{\nu=1}^N c_{\nu}(x+jh) \int_{x+Nh}^1 d_{\nu}(t) y(t) dt - \int_{x-Mh}^{x+Nh} K(x, t) y(t) dt = g(x+jh), \end{aligned}$$

$j = -M, -M+1, \dots, -1, 0, 1, \dots, N$.

Put

$$B_{\mu} = \int_0^{x-Mh} b_{\mu}(t) y(t) dt, \quad D_{\nu} = \int_{x+Nh}^1 d_{\nu}(t) y(t) dt,$$

$$Y_j = y(x+jh) - \int_{x-Mh}^{x+Nh} K(x+jh, t) y(t) dt - g(x+jh),$$

$$a_{j\mu} = a_{\mu}(x+jh), \quad c_{j\nu} = c_{\nu}(x+jh).$$

Then we have

$$(3) \quad \sum_{\mu=1}^M a_{j\mu} B_{\mu} + \sum_{\nu=1}^N c_{j\nu} D_{\nu} = Y_j$$

for $j = -M, \dots, -1, 0, 1, \dots, N$. Hence we have $M+N+1$ equations

and $M+N$ unknowns, $B_1, \dots, B_M, D_1, \dots, D_N$. If the integral equation (1) has a solution y , then all the $M+N+1$ equations in (3) must be satisfied. This means that the $M+N+1$ vectors

$$a_\mu = \begin{pmatrix} a_{-M,\mu} \\ \vdots \\ a_{N,\mu} \end{pmatrix}, \quad c_\nu = \begin{pmatrix} c_{-M,\nu} \\ \vdots \\ c_{N,\nu} \end{pmatrix}, \quad Y = \begin{pmatrix} Y_{-M} \\ \vdots \\ Y_N \end{pmatrix},$$

$\mu = 1, \dots, M$, $\nu = 1, \dots, N$, are linearly dependent. Hence we must have

$$(4) \quad \begin{vmatrix} a_{-M,1} & \dots & a_{-M,M} & c_{-M,1} & \dots & c_{-M,N} & Y_{-M} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ a_{0,1} & \dots & a_{0,M} & c_{0,1} & \dots & c_{0,N} & Y_0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ a_{N,1} & \dots & a_{N,M} & c_{N,1} & \dots & c_{N,N} & Y_N \end{vmatrix} = 0.$$

By expanding the determinant (4) after the last column we get a linear equation for the unknowns $y(x+jh)$, $j = -M, \dots, N$.

Let n define a grid G_n on the interval $[0, 1]$ with equidistant points, $x = i/n$, $i = 0, 1, \dots, n$, $h = 1/n$. We assume that $n \geq N+M$.

For $x = i/n$, $i = M, \dots, n-N$, we have by (4)

$$(5) \quad \begin{vmatrix} a_1\left(\frac{i-M}{n}\right) & \dots & c_N\left(\frac{i-M}{n}\right) & y\left(\frac{i-M}{n}\right) - \int_{(i-M)/n}^{(i+N)/n} K\left(\frac{i-M}{n}, t\right) y(t) dt - g\left(\frac{i-M}{n}\right) \\ \vdots & & \vdots & \vdots \\ a_1\left(\frac{i}{n}\right) & \dots & c_N\left(\frac{i}{n}\right) & y\left(\frac{i}{n}\right) - \int_{(i-M)/n}^{(i+N)/n} K\left(\frac{i}{n}, t\right) y(t) dt - g\left(\frac{i}{n}\right) \\ \vdots & & \vdots & \vdots \\ a_1\left(\frac{i+N}{n}\right) & \dots & c_N\left(\frac{i+N}{n}\right) & y\left(\frac{i+N}{n}\right) - \int_{(i-M)/n}^{(i+N)/n} K\left(\frac{i+N}{n}, t\right) y(t) dt - g\left(\frac{i+N}{n}\right) \end{vmatrix} = 0$$

For $x = i/n$, $i = 0, 1, \dots, M-1$, we have

$$(6) \quad y\left(\frac{j}{n}\right) - \int_0^{\frac{i+N}{n}} K\left(\frac{j}{n}, t\right) y(t) dt - \sum_{\nu=1}^N c_{\nu}\left(\frac{j}{n}\right) \int_{\frac{i+N}{n}}^1 d_{\nu}(t) y(t) dt = g\left(\frac{j}{n}\right),$$

for $j = 0, 1, \dots, i+N$.

Put

$$D_{\nu} = \int_{\frac{i+N}{n}}^1 d_{\nu}(t) y(t) dt, \quad \nu = 1, \dots, N,$$

$$c_{j\nu} = c_{\nu}\left(\frac{j}{n}\right), \quad j = i, \dots, i+N,$$

$$Y_j = y\left(\frac{j}{n}\right) - \int_0^{\frac{i+N}{n}} K\left(\frac{j}{n}, t\right) y(t) dt - g\left(\frac{j}{n}\right), \quad j = i, \dots, i+N.$$

We get the following linear system

$$(7) \quad \sum_{\nu=1}^N c_{j\nu} D_{\nu} = Y_j, \quad j = i, \dots, i+N.$$

Hence we have $N+1$ equations and N unknowns, $D_1, \dots, D_N$. As for equation (3) we now have

$$(8) \quad \begin{vmatrix} c_1\left(\frac{i}{n}\right) & \dots & c_N\left(\frac{i}{n}\right) & y\left(\frac{i}{n}\right) - \int_0^{\frac{i+N}{n}} K\left(\frac{i}{n}, t\right) y(t) dt - g\left(\frac{i}{n}\right) \\ \vdots & & \vdots & \vdots \\ c_1\left(\frac{i+N}{n}\right) & \dots & c_N\left(\frac{i+N}{n}\right) & y\left(\frac{i+N}{n}\right) - \int_0^{\frac{i+N}{n}} K\left(\frac{i+N}{n}, t\right) y(t) dt - g\left(\frac{i+N}{n}\right) \end{vmatrix} = 0,$$

for $i = 0, 1, \dots, M-1$.

Similarly for $x = i/n$, $i = n-N+1, \dots, n$, we have

$$(9) \quad y\left(\frac{j}{n}\right) - \sum_{\mu=1}^M a_{\mu}\left(\frac{j}{n}\right) \int_0^{\frac{i-M}{n}} b_{\mu}(t)y(t)dt - \int_{\frac{i-M}{n}}^1 K\left(\frac{j}{n}, t\right)y(t)dt = g\left(\frac{j}{n}\right),$$

for $j = i-M, \dots, n$.

Put

$$B_{\mu} = \int_0^{\frac{i-M}{n}} b_{\mu}(t)y(t)dt, \quad \mu = 1, \dots, M,$$

$$a_{j\mu} = a_{\mu}\left(\frac{j}{n}\right), \quad j = i-M, \dots, i,$$

$$Y_j = y\left(\frac{j}{n}\right) - \int_{\frac{i-M}{n}}^1 K\left(\frac{j}{n}, t\right)y(t)dt - g\left(\frac{j}{n}\right), \quad j = i-M, \dots, i.$$

We get the following linear system

$$(10) \quad \sum_{\mu=1}^M a_{j\mu} B_{\mu} = Y_j, \quad j = i-M, \dots, i,$$

with $M+1$ equations and M unknowns, $B_1, \dots, B_M$. Hence we must have

$$(11) \quad \begin{vmatrix} a_1\left(\frac{i-M}{n}\right) & \dots & a_M\left(\frac{i-M}{n}\right) & y\left(\frac{i-M}{n}\right) - \int_{\frac{i-M}{n}}^1 K\left(\frac{i-M}{n}, t\right)y(t)dt - g\left(\frac{i-M}{n}\right) \\ \vdots & & \vdots & \vdots \\ a_1\left(\frac{i}{n}\right) & & a_M\left(\frac{i}{n}\right) & y\left(\frac{i}{n}\right) - \int_{\frac{i-M}{n}}^1 K\left(\frac{i}{n}, t\right)y(t)dt - g\left(\frac{i}{n}\right) \end{vmatrix} = 0,$$

for $i = n-N+1, \dots, n$.

We call equations (5), (8), and (11) the determinant formulation of the integral equation (1).

Approximation of the integrals in (5), (8), and (11) by the trapezoidal rule on the grid G_n gives a linear system in which the matrix has a band structure with band width $M+N+1$.

We now assume that $M = N$. Then by determinant manipulation equation (5) can be written

$$\begin{aligned}
 & \left| \begin{array}{ccc}
 a_1(\frac{i-N}{n}) & \dots & c_N(\frac{i-N}{n}) y(\frac{i-N}{n}) - \sum_{v=1}^N \left(\frac{\frac{i}{n}}{\frac{i-N}{n}} \right) \left| \begin{array}{cc} d_v(t) & b_v(t) \\ a_v(\frac{i-N}{n}) & c_v(\frac{i-N}{n}) \end{array} \right| y(t) dt - \\
 & \qquad \qquad \qquad - g(\frac{i-N}{n}) \\
 \vdots & \qquad \qquad \qquad \vdots \\
 a_1(\frac{i-1}{n}) & \dots & c_N(\frac{i-1}{n}) y(\frac{i-1}{n}) - \sum_{v=1}^N \left(\frac{\frac{i}{n}}{\frac{i-1}{n}} \right) \left| \begin{array}{cc} d_v(t) & b_v(t) \\ a_v(\frac{i-1}{n}) & c_v(\frac{i-1}{n}) \end{array} \right| y(t) dt - \\
 & \qquad \qquad \qquad - g(\frac{i-1}{n}) \\
 (12) & a_1(\frac{i}{n}) \dots c_N(\frac{i}{n}) y(\frac{i}{n}) - g(\frac{i}{n}) \\
 & a_1(\frac{i+1}{n}) \dots c_N(\frac{i+1}{n}) y(\frac{i+1}{n}) - \sum_{v=1}^N \left(\frac{\frac{i+1}{n}}{\frac{i}{n}} \right) \left| \begin{array}{cc} a_v(\frac{i+1}{n}) & c_v(\frac{i+1}{n}) \\ d_v(t) & b_v(t) \end{array} \right| y(t) dt - \\
 & \qquad \qquad \qquad - g(\frac{i+1}{n}) \\
 \vdots & \qquad \qquad \qquad \vdots \\
 & a_1(\frac{i+N}{n}) \dots c_N(\frac{i+N}{n}) y(\frac{i+N}{n}) - \sum_{v=1}^N \left(\frac{\frac{i+N}{n}}{\frac{i}{n}} \right) \left| \begin{array}{cc} a_v(\frac{i+N}{n}) & c_v(\frac{i+N}{n}) \\ d_v(t) & b_v(t) \end{array} \right| y(t) dt - \\
 & \qquad \qquad \qquad - g(\frac{i+N}{n})
 \end{array} \right| = 0
 \end{aligned}$$

for $i = N, \dots, n-N$.

Similarly for equations (8) and (11) we have

$$\begin{aligned}
 & \left| \begin{array}{ccc} c_1(\frac{i}{n}) & \dots & c_N(\frac{i}{n}) & y(\frac{i}{n}) - g(\frac{i}{n}) \\ c_1(\frac{i+1}{n}) & \dots & c_N(\frac{i+1}{n}) & y(\frac{i+1}{n}) - \sum_{v=1}^N \int_{\frac{i}{n}}^{\frac{i+1}{n}} \begin{vmatrix} a_v(\frac{i+1}{n}) & c_v(\frac{i+1}{n}) \\ d_v(t) & b_v(t) \end{vmatrix} y(t) dt - \\ & & & - g(\frac{i+1}{n}) \\ \vdots & & \vdots & \vdots \\ c_1(\frac{i+N}{n}) & \dots & c_N(\frac{i+N}{n}) & y(\frac{i+N}{n}) - \sum_{v=1}^N \int_{\frac{i}{n}}^{\frac{i+N}{n}} \begin{vmatrix} a_v(\frac{i+N}{n}) & c_v(\frac{i+N}{n}) \\ d_v(t) & b_v(t) \end{vmatrix} y(t) dt - \\ & & & - g(\frac{i+N}{n}) \end{array} \right| = 0
 \end{aligned}
 \quad (13)$$

for $i = 0, 1, \dots, N-1$, and

$$\begin{aligned}
 & \left| \begin{array}{ccc} a_1(\frac{i-N}{n}) & \dots & a_N(\frac{i-N}{n}) & y(\frac{i-N}{n}) - \sum_{v=1}^N \int_{\frac{i-N}{n}}^{\frac{i}{n}} \begin{vmatrix} d_v(t) & b_v(t) \\ a_v(\frac{i-N}{n}) & c_v(\frac{i-N}{n}) \end{vmatrix} y(t) dt - \\ & & & - g(\frac{i-N}{n}) \\ \vdots & & \vdots & \vdots \\ a_1(\frac{i-1}{n}) & \dots & a_N(\frac{i-1}{n}) & y(\frac{i-1}{n}) - \sum_{v=1}^N \int_{\frac{i-1}{n}}^{\frac{i}{n}} \begin{vmatrix} d_v(t) & b_v(t) \\ a_v(\frac{i-1}{n}) & c_v(\frac{i-1}{n}) \end{vmatrix} y(t) dt - \\ & & & - g(\frac{i-1}{n}) \\ a_1(\frac{i}{n}) & \dots & a_N(\frac{i}{n}) & y(\frac{i}{n}) - g(\frac{i}{n}) \end{array} \right| = 0
 \end{aligned}
 \quad (14)$$

for $i = n-N+1, \dots, n$.

Approximation of the integrals in (12), (13), and (14) by the trapezoidal rule on the grid G_n gives a linear system in which the matrix has a band structure with band width $2N+1$.

$$(18) \quad \left| \begin{array}{ccc} c_1(\frac{i}{n}) & \dots & c_N(\frac{i}{n}) \quad y(\frac{i}{n}) - \int_0^{\frac{i+N}{n}} K(\frac{i}{n}, t) f(t, y) dt - g(\frac{i}{n}) \\ \vdots & & \vdots \\ c_1(\frac{i+N}{n}) & \dots & c_N(\frac{i+N}{n}) \quad y(\frac{i+N}{n}) - \int_0^{\frac{i+N}{n}} K(\frac{i+N}{n}, t) f(t, y) dt - g(\frac{i+N}{n}) \end{array} \right| = 0,$$

for $i = 0, 1, \dots, M-1$, and

$$(19) \quad \left| \begin{array}{ccc} a_1(\frac{i-M}{n}) & \dots & a_M(\frac{i-M}{n}) \quad y(\frac{i-M}{n}) - \int_{\frac{i-M}{n}}^1 K(\frac{i-M}{n}, t) f(t, y) dt - g(\frac{i-M}{n}) \\ \vdots & & \vdots \\ a_1(\frac{i}{n}) & \dots & a_M(\frac{i}{n}) \quad y(\frac{i}{n}) - \int_{\frac{i-M}{n}}^1 K(\frac{i}{n}, t) f(t, y) dt - g(\frac{i}{n}) \end{array} \right| = 0,$$

for $i = n-N+1, \dots, n$.

Put

$$(20) \quad D_i(y) = \begin{cases} \begin{vmatrix} c_1(\frac{i}{n}) & \dots & c_N(\frac{i}{n}) & y(\frac{i}{n}) \\ \vdots & & \vdots & \vdots \\ c_1(\frac{i+N}{n}) & \dots & c_N(\frac{i+N}{n}) & y(\frac{i+N}{n}) \end{vmatrix}, & i = 0, 1, \dots, M-1, \\ \\ \begin{vmatrix} a_1(\frac{i-M}{n}) & \dots & c_N(\frac{i-M}{n}) & y(\frac{i-M}{n}) \\ \vdots & & \vdots & \vdots \\ a_1(\frac{i+N}{n}) & \dots & c_N(\frac{i+N}{n}) & y(\frac{i+N}{n}) \end{vmatrix}, & i = M, \dots, n-N, \\ \\ \begin{vmatrix} a_1(\frac{i-M}{n}) & \dots & a_M(\frac{i-M}{n}) & y(\frac{i-M}{n}) \\ \vdots & & \vdots & \vdots \\ a_1(\frac{i}{n}) & \dots & a_M(\frac{i}{n}) & y(\frac{i}{n}) \end{vmatrix}, & i = n-N+1, \dots, n, \end{cases}$$

and

$$(21) \quad D_i(K, t) = \begin{cases} \begin{vmatrix} c_1(\frac{i}{n}) & \dots & c_N(\frac{i}{n}) & K(\frac{i}{n}, t) \\ \vdots & & \vdots & \vdots \\ c_1(\frac{i+N}{n}) & \dots & c_N(\frac{i+N}{n}) & K(\frac{i+N}{n}, t) \end{vmatrix}, & i = 0, \dots, M-1, \\ \\ \begin{vmatrix} a_1(\frac{i-M}{n}) & \dots & c_N(\frac{i-M}{n}) & K(\frac{i-M}{n}, t) \\ \vdots & & \vdots & \vdots \\ a_1(\frac{i+N}{n}) & \dots & c_N(\frac{i+N}{n}) & K(\frac{i+N}{n}, t) \end{vmatrix}, & i = M, \dots, n-N, \\ \\ \begin{vmatrix} a_1(\frac{i-M}{n}) & \dots & a_M(\frac{i-M}{n}) & K(\frac{i-M}{n}, t) \\ \vdots & & \vdots & \vdots \\ a_1(\frac{i}{n}) & \dots & a_M(\frac{i}{n}) & K(\frac{i}{n}, t) \end{vmatrix}, & i = n-N+1, \dots, n. \end{cases}$$

Then equations (17), (18), and (19) can be written

$$(22) \quad D_i(y) - \int_{\max(0, \frac{i-M}{n})}^{\min(1, \frac{i+N}{n})} D_i(K, t) f(t, y(t)) dt = D_i(g),$$

for $i = 0, 1, \dots, n$. Newton's method for equation (22) is

$$(23) \quad \left\{ \begin{array}{l} D_i(u) - \int_{\max(0, \frac{i-M}{n})}^{\min(1, \frac{i+N}{n})} D_i(K, t) f'_y(t, y(t)) u(t) dt = \\ \\ = - \left[D_i(y) - \int_{\max(0, \frac{i-M}{n})}^{\min(1, \frac{i+N}{n})} D_i(K, t) f(t, y(t)) dt - D_i(g) \right], \\ \\ \text{for } i = 0, 1, \dots, n, \\ \\ y := y + u. \end{array} \right.$$

Approximation of the integrals in (23), by the trapezoidal rule on the grid G_n , gives in each step a system of linear algebraic equations for the approximate values of u at the points i/n , $i = 0, 1, \dots, n$. The matrices of these systems are band matrices with band width $M+N+1$.

The results of this section can immediately be generalized to integral equations of Urysohn type

$$y(x) - \int_0^1 K(x, t, y(t)) dt = g(x),$$

where the kernel K is a nonlinear semidegenerate kernel defined by

$$K(x, t, y) = \begin{cases} \sum_{\mu=1}^M a_{\mu}(x) b_{\mu}(t, y), & x > t, \\ \sum_{\nu=1}^N c_{\nu}(x) d_{\nu}(t, y), & x < t. \end{cases}$$

4. NEWTON-LIKE METHODS

Newton's method for the integral equation (15) is

$$(24) \quad \begin{cases} u(x) - \int_0^1 K(x,t) f'_y(t, y(t)) u(t) dt = \\ = - \left[y(x) - \int_0^1 K(x,t) f(t, y(t)) dt - g(x) \right], \\ y := y + u. \end{cases}$$

Let m, n , and r be positive integers, $n = rm$. Then m and n define two grids, G_m and G_n , on $[0, 1]$ with equidistant points. Approximation of the integral of the left hand side of (24) by the trapezoidal rule on the grid G_m and the integral of the right hand side by the same rule but on the finer grid G_n gives

$$(25) \quad \begin{cases} u(x) - \sum_{j=0}^m w_j K(x, \frac{j}{m}) f'_y(\frac{j}{m}, y(\frac{j}{m})) u(\frac{j}{m}) = \\ = - \left[y(x) - \sum_{k=0}^n w_k K(x, \frac{k}{n}) f(\frac{k}{n}, y(\frac{k}{n})) - g(x) \right], \\ y := y + u. \end{cases}$$

Equation (25) defines a Newton-like method (see Aulin [1]) for the sum equation

$$(26) \quad y(x) - \sum_{k=0}^n w_k K(x, \frac{k}{n}) f(\frac{k}{n}, y(\frac{k}{n})) = g(x).$$

Since

$$(27) \quad \sum_{j=0}^m w_j K\left(\frac{p}{m}, \frac{j}{m}\right) f_j =$$

$$= \begin{cases} 0 + \sum_{v=1}^N c_v(0) \sum_{j=0}^m w_j d_v\left(\frac{j}{m}\right) f_j, & p = 0, \\ \sum_{\mu=1}^M a_{\mu}\left(\frac{p}{m}\right) \sum_{j=0}^p w_j b_{\mu}\left(\frac{j}{m}\right) f_j + \sum_{v=1}^N c_v\left(\frac{p}{m}\right) \sum_{j=p}^m w_j d_v\left(\frac{j}{m}\right) f_j, & 1 \leq p \leq m-1, \\ \sum_{\mu=1}^M a_{\mu}(1) \sum_{j=0}^m w_j b_{\mu}\left(\frac{j}{m}\right) f_j + 0, & p = m, \end{cases}$$

where in each sum the weights w_j are those of the trapezoidal rule, we obtain the following discrete determinant formulation of equation (25)

$$(28) \quad D_i(u) - \sum_{j=\max(0, i-M)}^{\min(m, i+N)} w_j D_i\left(K, \frac{j}{m}\right) f'_y\left(\frac{j}{m}, y\left(\frac{j}{m}\right)\right) u\left(\frac{j}{m}\right) =$$

$$= - \left[D_i(y) - \sum_{k=\max(0, r(i-M))}^{\min(n, r(i+N))} w_k D_i\left(K, \frac{k}{n}\right) f\left(\frac{k}{n}, y\left(\frac{k}{n}\right)\right) - D_i(g) \right],$$

for $i = 0, 1, \dots, m$, where the determinants D_i are defined by (20) and (21). If we solve equation (28) for $u(i/m)$, $i = 0, 1, \dots, m$, then equation (25) is satisfied at the points $x = i/m$, $i = 0, 1, \dots, m$.

To define equation (25) for intermediate values let $x = \frac{i}{m} + \frac{q}{n}$, $q = 1, \dots, r-1$. Approximation by the trapezoidal rule gives

$$\begin{aligned}
& \int_0^1 K\left(\frac{i}{m} + \frac{q}{n}, t\right) f'_y(t, y(t)) u(t) dt = \\
& = \int_0^{\frac{i}{m}} K\left(\frac{i}{m} + \frac{q}{n}, t\right) f'_y(t, y(t)) u(t) dt + \int_{\frac{i}{m}}^{\frac{i+1}{m}} K\left(\frac{i}{m} + \frac{q}{n}, t\right) f'_y(t, y(t)) u(t) dt + \\
& + \int_{\frac{i+1}{m}}^1 K\left(\frac{i}{m} + \frac{q}{n}, t\right) f'_y(t, y(t)) u(t) dt \cong \\
& \cong \sum_{\mu=1}^M a_{\mu}\left(\frac{i}{m} + \frac{q}{n}\right) \sum_{j=0}^i w_j b_{\mu}\left(\frac{j}{m}\right) f'_y\left(\frac{j}{m}, y\left(\frac{j}{m}\right)\right) u\left(\frac{j}{m}\right) + \\
& + \frac{1}{2m} \left[\sum_{\mu=1}^M a_{\mu}\left(\frac{i}{m} + \frac{q}{n}\right) b_{\mu}\left(\frac{i}{m}\right) f'_y\left(\frac{i}{m}, y\left(\frac{i}{m}\right)\right) u\left(\frac{i}{m}\right) + \right. \\
& + \left. \sum_{\nu=1}^N c_{\nu}\left(\frac{i}{m} + \frac{q}{n}\right) d_{\nu}\left(\frac{i+1}{m}\right) f'_y\left(\frac{i+1}{m}, y\left(\frac{i+1}{m}\right)\right) u\left(\frac{i+1}{m}\right) \right] + \\
& + \sum_{\nu=1}^N c_{\nu}\left(\frac{i}{m} + \frac{q}{n}\right) \sum_{j=i+1}^m w_j d_{\nu}\left(\frac{j}{m}\right) f'_y\left(\frac{j}{m}, y\left(\frac{j}{m}\right)\right) u\left(\frac{j}{m}\right),
\end{aligned}$$

where in each sum the weights are those of the trapezoidal rule. Hence the approximation of the integral of the left hand side of equation (24) can be written

$$\begin{aligned}
(29) \quad & \sum_{j=0}^m w_j K\left(\frac{i}{m} + \frac{q}{n}, \frac{j}{m}\right) f'_y\left(\frac{j}{m}, y\left(\frac{j}{m}\right)\right) u\left(\frac{j}{m}\right) = \\
& = \sum_{\mu=1}^M a_{\mu}\left(\frac{i}{m} + \frac{q}{n}\right) \sum_{j=0}^i \bar{w}_j b_{\mu}\left(\frac{j}{m}\right) f'_y\left(\frac{j}{m}, y\left(\frac{j}{m}\right)\right) u\left(\frac{j}{m}\right) + \\
& + \sum_{\nu=1}^N c_{\nu}\left(\frac{i}{m} + \frac{q}{n}\right) \sum_{j=i+1}^m \bar{w}_j d_{\nu}\left(\frac{j}{m}\right) f'_y\left(\frac{j}{m}, y\left(\frac{j}{m}\right)\right) u\left(\frac{j}{m}\right),
\end{aligned}$$

where $\bar{w}_0 = \bar{w}_m = \frac{1}{2m}$, $\bar{w}_1 = \dots = \bar{w}_{m-1} = \frac{1}{m}$.

Equation (28) can be written as three determinants

$$(30) \quad \begin{vmatrix} c_1(\frac{i}{m}) & \dots & c_N(\frac{i}{m}) & y(\frac{i}{m}) + u(\frac{i}{m}) - K_0(\frac{i}{m}) - g(\frac{i}{m}) \\ \vdots & & \vdots & \vdots \\ c_1(\frac{i+N}{m}) & \dots & c_N(\frac{i+N}{m}) & y(\frac{i+N}{m}) + u(\frac{i+N}{m}) - K_0(\frac{i+N}{m}) - g(\frac{i+N}{m}) \end{vmatrix} = 0,$$

for $i = 0, 1, \dots, m-1$,

$$(31) \quad \begin{vmatrix} a_1(\frac{i-M}{m}) & \dots & c_N(\frac{i-M}{m}) & y(\frac{i-M}{m}) + u(\frac{i-M}{m}) - K(\frac{i-M}{m}) - g(\frac{i-M}{m}) \\ \vdots & & \vdots & \vdots \\ a_1(\frac{i}{m}) & \dots & c_N(\frac{i}{m}) & y(\frac{i}{m}) + u(\frac{i}{m}) - K(\frac{i}{m}) - g(\frac{i}{m}) \\ \vdots & & \vdots & \vdots \\ a_1(\frac{i+N}{m}) & \dots & c_N(\frac{i+N}{m}) & y(\frac{i+N}{m}) + u(\frac{i+N}{m}) - K(\frac{i+N}{m}) - g(\frac{i+N}{m}) \end{vmatrix} = 0,$$

for $i = m, \dots, m-N$, and

$$(32) \quad \begin{vmatrix} a_1(\frac{i-M}{m}) & \dots & a_M(\frac{i-M}{m}) & y(\frac{i-M}{m}) + u(\frac{i-M}{m}) - K_1(\frac{i-M}{m}) - g(\frac{i-M}{m}) \\ \vdots & & \vdots & \vdots \\ a_1(\frac{i}{m}) & \dots & a_M(\frac{i}{m}) & y(\frac{i}{m}) + u(\frac{i}{m}) - K_1(\frac{i}{m}) - g(\frac{i}{m}) \end{vmatrix} = 0,$$

for $i = m - N + 1$, where

$$K_0(\frac{i}{m}) = \sum_{j=0}^{i+N} w_j K(\frac{i}{m}, \frac{j}{m}) f'_y(\frac{j}{m}, y(\frac{j}{m})) u(\frac{j}{m}) + \sum_{k=0}^{r(i+N)} w_k K(\frac{i}{m}, \frac{k}{n}) f(\frac{k}{n}, y(\frac{k}{n})),$$

$$K(\frac{i}{m}) = \sum_{j=i-M}^{i+N} w_j K(\frac{i}{m}, \frac{j}{m}) f'_y(\frac{j}{m}, y(\frac{j}{m})) u(\frac{j}{m}) + \sum_{k=r(i-M)}^{r(i+N)} w_k K(\frac{i}{m}, \frac{k}{n}) f(\frac{k}{n}, y(\frac{k}{n})),$$

$$K_1(\frac{i}{m}) = \sum_{j=i-M}^m w_j K(\frac{i}{m}, \frac{j}{m}) f'_y(\frac{j}{m}, y(\frac{j}{m})) u(\frac{j}{m}) + \sum_{k=r(i-M)}^n w_k K(\frac{i}{m}, \frac{k}{n}) f(\frac{k}{n}, y(\frac{k}{n})).$$

Now intermediate values $u(\frac{i}{m} + \frac{q}{n})$, $q = 1, \dots, r-1$, can be found by replacing the variable $\frac{i}{m}$ with $\frac{i}{m} + \frac{q}{n}$ in (30), (31), and with $\frac{i-1}{m} + \frac{q}{n}$ in (32). Hence we have

$$(33) \quad \begin{vmatrix} c_1(\frac{i+q}{m+n}) & \dots & c_N(\frac{i+q}{m+n}) & y(\frac{i+q}{m+n}) + u(\frac{i+q}{m+n}) - K_0(\frac{i+q}{m+n}) - g(\frac{i+q}{m+n}) \\ \vdots & & \vdots & \vdots \\ c_1(\frac{i+N}{m}) & \dots & c_N(\frac{i+N}{m}) & y(\frac{i+N}{m}) + u(\frac{i+N}{m}) - K_0(\frac{i+N}{m}) - g(\frac{i+N}{m}) \end{vmatrix} = 0,$$

for $i = 0, 1, \dots, M-1$,

$$(34) \quad \begin{vmatrix} a_1(\frac{i-M}{m}) & \dots & c_N(\frac{i-M}{m}) & y(\frac{i-M}{m}) + u(\frac{i-M}{m}) - K(\frac{i-M}{m}) - g(\frac{i-M}{m}) \\ \vdots & & \vdots & \vdots \\ a_1(\frac{i+q}{m+n}) & \dots & c_N(\frac{i+q}{m+n}) & y(\frac{i+q}{m+n}) + u(\frac{i+q}{m+n}) - K(\frac{i+q}{m+n}) - g(\frac{i+q}{m+n}) \\ \vdots & & \vdots & \vdots \\ a_1(\frac{i+N}{m}) & \dots & c_N(\frac{i+N}{m}) & y(\frac{i+N}{m}) + u(\frac{i+N}{m}) - K(\frac{i+N}{m}) - g(\frac{i+N}{m}) \end{vmatrix} = 0,$$

for $i = M, \dots, m-N$,

$$(35) \quad \begin{vmatrix} a_1(\frac{i-M}{m}) & \dots & a_M(\frac{i-M}{m}) & y(\frac{i-M}{m}) + u(\frac{i-M}{m}) - K_1(\frac{i-M}{m}) - g(\frac{i-M}{m}) \\ \vdots & & \vdots & \vdots \\ a_1(\frac{i-1}{m} + \frac{q}{n}) & \dots & a_M(\frac{i-1}{m} + \frac{q}{n}) & y(\frac{i-1}{m} + \frac{q}{n}) + u(\frac{i-1}{m} + \frac{q}{n}) - K_1(\frac{i-1}{m} + \frac{q}{n}) - g(\frac{i-1}{m} + \frac{q}{n}) \end{vmatrix} = 0,$$

for $i = m-N+2, \dots, m$.

Equations (33), (34), and (35) define $u(\frac{i}{m} + \frac{q}{n})$ explicitly

for $q = 1, \dots, r-1$, $i = 0, 1, \dots, m-1$.

Remark. To compute intermediate values we could have replaced any variable $\frac{i+j}{m}$ with $\frac{i+j}{m} + \frac{q}{n}$ as long as $\frac{i+j}{m} + \frac{q}{n}$ is in the allowed interval, i.e. $[0, \frac{i+N}{m}]$ in equation (30), $[\frac{i-M}{m}, \frac{i+N}{m}]$ in equation (31), and $[\frac{i-M}{m}, 1]$ in equation (32). Moreover if $M = N = 1$ it is better to consider the points $\frac{i}{m}$, $\frac{i}{m} + \frac{q}{n}$, $\frac{i+1}{m}$ instead of

$$\frac{i-1}{m}, \frac{i}{m} + \frac{q}{n}, \frac{i+1}{m} \quad (\text{see Aulin } [2], [3]).$$

Convergence results for the Newton-like method (25) can be found in Aulin [1] theorem 5.4 and 5.5, and for two-point boundary value problems in Aulin [2].

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